2. Mochizuki

Assume X : proj. surface $/\mathbb{C}$ $\quad$ H : ample line bdle

$M_H(y)$: moduli space of semistable torsion free sheaves $\quad$ $rk=2$

<u>Th</u>, (T. Mochizuki)

$$\int_{[M_H(y)]^{vir}} \exp(\mu(\alpha z + p x)) = \sum_{\mathfrak{z}_1} SW(\tilde{\mathfrak{z}}_1) \operatorname*{Res}_{a=\infty} \tilde{\mathcal{A}}(\mathfrak{z}_1, y ; a)\,da \qquad (\tilde{\mathfrak{z}}_1 = 2\mathfrak{z}_1 - K_X)$$

$y = (2, \mathfrak{z}, n)$

under some assumption on $\mathfrak{z}$ (which holds if we replace $\mathfrak{z} + 2NH$)
$N \gg 0$

Here $\tilde{\mathcal{A}}$ is an explicit integral over Hilbert scheme of points on X :

$$\tilde{\mathcal{A}}(\mathfrak{z}_1, y ; a) = 2^{1-\chi(y)}$$

$$\mathfrak{z}_2 = \mathfrak{z} - \mathfrak{z}_1 \qquad \times \sum_{n_1+n_2 = n - (\mathfrak{z}_1, \mathfrak{z}_2)} \frac{1}{a^{n_1+n_2-p_g}} \int_{X^{[n_1]} \times X^{[n_2]}} \frac{\exp \mu(\nu z + p x) \cap e(H^*(E_1 \oplus E_2))}{e(-Ext^*(E_1, E_2) - Ext^*(E_2, E_1))}$$

with
$$\begin{cases} E_1 = I_1 \otimes e^{\mathfrak{z}_1} \\ E_2 = I_2 \otimes e^{\mathfrak{z}_2 + 2a} \end{cases}$$

(technique of the proof)

▷ Master space

$Q \circlearrowleft G \qquad L_+, L_- :$ ample line b'dles

$$\Longrightarrow \quad M_\pm := Q /\!\!/_{L_\pm} G$$

Want to relate $\int_{M_+} \alpha$ and $\int_{M_-} \alpha$.

Consider the master space $\mathcal{M} = \mathbb{P}(L_+^{-1} \oplus L_-^{-1}) /\!\!/ G$ with a natural $\mathbb{C}^*$-action

natural polarization $\qquad [z_+ : z_-] \mapsto [e^a z_+ : z_-]$

$$\mathcal{M}^{\mathbb{C}^*} = M_+ \sqcup M_- \sqcup \bigcup_p \mathcal{F}_p$$

"exceptional fixed pts

a : generator of $H^*_{\mathbb{C}^*}(pt) = H^*(\mathbb{CP}^\infty) = \mathbb{C}[a]$

Then $\displaystyle 0 = \int_{\mathcal{M}} \alpha a \Big|_{a=0} \overset{\text{fixed pt formula}}{=} \text{Coeff. of } a^0 \text{ in } \int_{\mathcal{M}^{\mathbb{C}^*}} \frac{\alpha a}{e(\text{Normal b'dle})}$

$$= \int_{M_+} \alpha - \int_{M_-} \alpha + \operatorname*{Res}_{a=0} \sum_p \int_{\mathcal{F}_p} \frac{\alpha}{e(N_{\mathcal{F}_p/\mathcal{M}})}$$

Roughly $M_\pm$: moduli space of semistable pairs (E, s) $\left(\begin{array}{l} E : \text{rk } 2 \text{ sheaf} \\ s \in H^0(E) \end{array} \right.$

with appropriate polarization

(polarization for E & pol. for s
$\Rightarrow$ 1 param. family of polari.

$\rightsquigarrow$ · $M_+ \longrightarrow M_H$ projective bundle

· $M_- = \phi$

· exceptional fixed pts = direct sum of rk 1 pairs
$\simeq$ line bdles $\otimes$ ideal sheaves

$\Longrightarrow$ SW & Hilbert schemes

3 Universality

$\longrightarrow$ omitted

It is enough to compute $\tilde{A} = \sum \int_{X^{[n_1]} \times X^{[n_2]}} \cdots$ for $X =$ projective toric surface.

$X \curvearrowleft T^2$

$X^{[n]} \curvearrowleft T^2$ fixed pts $= T^2$-inv. subschemes supported at X^T

no interaction between p_i & p_j
except contribution from line bdles ξ_1 & ξ_2

$\longrightarrow$ enough to work on $\mathbb{C}^2$ $\rightsquigarrow$ instanton counting!

4. partition function for $N=2$ $SU(2)$ SUSY YM with one fund. matter

(after Nekrasov)

$M(n) = M(2,n)$: moduli sp. of framed sheaves (E,φ) on $\mathbb{P}^2 = \mathbb{C}^2 \cup \ell_\infty$

$\varphi : E|_{\ell_\infty} \cong \mathcal{O}_{\ell_\infty}^{\oplus 2}$

$\curvearrowleft T^3$ $\quad \mathrm{Lie}\, T^3 = \mathbb{C}\varepsilon_1 \oplus \mathbb{C}\varepsilon_2 \oplus \mathbb{C}a$ ← change of frame

matter bdle $\quad \mathcal{V}_{(E,\varphi)} = H^1(E(-\ell_\infty)) \otimes \underbrace{K_{\mathbb{C}^2}^{1/2}}_{= e^{-\varepsilon_1+\varepsilon_2/2}}$ $\quad$ (kind of pairs)

$\curvearrowleft S^1$ multiplication

$\mathrm{Lie}\, S^1 = \mathbb{C}m$ $\quad$ (matter)

instanton partition function

$$Z^{in}(\varepsilon_1,\varepsilon_2,a,m,\Lambda) \equiv \sum_n \Lambda^{3n} \int_{M(n)} e(\mathcal{V} \otimes e^m)$$

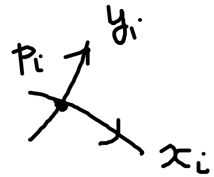
This is defined by the fixed pt formula

$$= \sum_n \Lambda^{3n} \sum_{p \in M(n)^{T^3}} \frac{e(\mathcal{V}\otimes e^m)|_p}{e(T_p M(n))}$$

fixed pts $\quad \coprod_{n=n_1+n_2} \mathrm{Hilb}^{n_1}(\mathbb{C}^2)^{T^2} \times \mathrm{Hilb}^{n_2}(\mathbb{C}^2)^{T^2}$

monomial ideals $\leftrightarrow$ Young diagrams

Let $X^{T^2} = \{p_1, \cdots, p_\chi\}$



torus equiv coord

$(w(x_i), w(y_i))$: weights ($m_i\varepsilon_1 + n_i\varepsilon_2$ $(m_i, n_i) \in \mathbb{Z}$)

Let $\mathcal{B}(\xi, \xi; a) := \sum_n \Lambda^{\overset{4n-(\xi^2)-3\chi_h(X)}{\dim_{\mathbb{C}} M_H(y)}} \tilde{\mathcal{A}}(\xi_1, \overset{(2,\xi,n)}{\overset{\shortparallel}{y}}; a)$

Prop $\mathcal{B}(\xi_1, \xi : a) = (\text{explicit func}) \leftarrow$ contribution from $\xi_1, \xi_2 = \xi - \xi_1$

$$\times \lim_{\varepsilon_1, \varepsilon_2 \to 0} \prod_{i=1}^{\chi} Z^{in}\left(w(x_i), w(y_i), \iota_{p_i}^*\left(\frac{\xi_2 - \xi_1}{2}\right) + a, \iota_{p_i}^*\left(\frac{\xi - K_X}{2}\right) + a, \frac{\Lambda^{4/3}}{a^{1/6}} e^{\iota_{p_i}^*\left(\frac{\alpha z + \beta x}{3}\right)}\right)$$

$\iota_{p_i} : \{p_i\} \hookrightarrow X$

Prop $\log Z^{in} = \frac{1}{\varepsilon_1\varepsilon_2}\left(F_0^{in} + (\varepsilon_1 + \varepsilon_2)\underbrace{H^{in}}_{\overset{\shortparallel}{0}} + \varepsilon_1\varepsilon_2 A^{in} + \frac{\varepsilon_1^2 + \varepsilon_2^2}{3} B^{in} + \text{higher order in } (\varepsilon_1, \varepsilon_2)\right)$

Th 4.5 $\quad \mathcal{B}(\mathfrak{z}_1, \mathfrak{z}; a) = (\text{explicit})$

$$\times \exp\Big[\frac{1}{3}\frac{\partial F_0^{\mathrm{in}}}{\partial \log\Lambda}x + \frac{1}{8}\frac{\partial^2 F_0^{\mathrm{in}}}{\partial a^2}(\mathfrak{z}_2-\mathfrak{z}_1)^2 + \frac{1}{4}\frac{\partial^2 F_0^{\mathrm{in}}}{\partial a\,\partial m}(\mathfrak{z}_2-\mathfrak{z}_1, \mathfrak{z}-K_X) + \frac{1}{8}\frac{\partial^2 F_0^{\mathrm{in}}}{\partial m^2}(\mathfrak{z}-K_X)^2$$

$$+\frac{1}{6}\frac{\partial^2 F_0^{\mathrm{in}}}{\partial a\,\partial \log\Lambda}(\mathfrak{z}_2-\mathfrak{z}_1.\alpha)z + \frac{1}{6}\frac{\partial^2 F_0^{\mathrm{in}}}{\partial m\,\partial \log\Lambda}(\mathfrak{z}-K_X,\alpha)z + \frac{1}{18}\frac{\partial^2 F_0^{\mathrm{in}}}{(\partial \log\Lambda)^2}(\alpha^2)z^2$$

$$+\chi(X)A^{\mathrm{in}} + \sigma(X)B^{\mathrm{in}}\Big] \quad \text{evaluated at } m=a$$

e.g. $$\sum_i \frac{1}{w(x_i)w(y_i)} \times \frac{1}{2}\frac{\partial^2 F_0^{\mathrm{in}}}{\partial \log\Lambda^2}\iota_{p_i}^*\left(\frac{\alpha z + px}{3}\right)^2 \underset{\varepsilon_1,\varepsilon_2=0}{\rightsquigarrow} \frac{1}{18}\frac{\partial^2 F_0^{\mathrm{in}}}{(\partial \log\Lambda)^2}(\alpha^2)z^2$$